Complex roots of unity & polynomial equations

Exercise 3.9

Solve the following equations on $\mathbb{C}$

a. $z^2 = i$

b. $z^2 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

c. $z^3 + 2 - 2i = 0$

d. $z^3 + 4 - 4\sqrt{3}i = 0$

e. $z^4 = -7 + 24i$

f. $z^4 = -7 + 4\sqrt{2}i$

g. $z^8 = \sqrt{3} + i$

h. $z^7 - 2iz^4 - iz^3 - 2 = 0$

i. $z^6 + iz^3 + i - 1 = 0$

Solution Exercise 3.9

a. $z^2 = e^{\frac{1}{2}\pi i + k2\pi i}$

$z = e^{\frac{1}{4}\pi i + k\pi i}$

$z = \frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2} \vee z = z = -\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}$

b.

$$z^2 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$$

$$z^2 = e^{\frac{1}{4}\pi i + k2\pi i}$$

$$z = e^{\frac{1}{8}\pi i + k\pi i}$$

$$z = \cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi \vee z = \cos\frac{9}{8}\pi + i\sin\frac{9}{8}\pi$$

$$z = \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}} \vee z = -\frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}$$

note: the exact answers were found using the half angle formulae of sin and cos, there are many other ways to compute an exact answer void of trigonometric functions.

c. $z^3 = -2 + 2i$

$z^3 = 8^{\frac{1}{6}} e^{\frac{3}{4}\pi i + k2\pi i}$

$z = \sqrt{2} e^{\frac{1}{4}\pi i + k\frac{2}{3}\pi i}$

$z = 1 + i \vee z = -\frac{1}{2} + \frac{1}{2}\sqrt{3} - \left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)i \vee z = -\frac{1}{2} - \frac{1}{2}\sqrt{3} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{3}\right)i$

d. $z^3 = -4 + 4\sqrt{3}i$

$z^3 = 8e^{\arctan_2\left(4\sqrt{3},-4\right)i+k2\pi i}$

$z = 2e^{\frac{2}{9}\pi i+k\frac{6}{9}\pi i}$

$z = 2\cos\left(\frac{2}{9}\pi\right) + 2i\sin\left(\frac{2}{9}\pi\right)$

$\vee\ z = 2\cos\left(\frac{8}{9}\pi\right) + 2i\sin\left(\frac{8}{9}\pi\right)$

$\vee\ z = 2\cos\left(\frac{14}{9}\pi\right) + 2i\sin\left(\frac{14}{9}\pi\right)$

note that $\arctan_2\left(4\sqrt{3},-4\right) = \frac{2}{3}\pi \neq \arctan\left(\frac{4\sqrt{3}}{-4}\right)$ *as* $\arctan_2$ *returns the angle in the right quadrant.*

e. $z^4 = -7 + 24i$

because $\arctan_2(24,-7)$ does not have an exact solutions, we will use substitution

$z^2 = u = a + bi$ with $a, b \in \mathbb{R}$

$u^2 = -7 + 24i$

$a^2 + 2abi - b^2 = -7 + 24i$

$\Rightarrow a^2 - b^2 = -7 \wedge 2ab = 24$

$a = \frac{12}{b}$

$144b^{-2} - b^2 = -7$

$b^4 - 7b^2 - 144 = 0$

$(b^2 - 16)(b^2 + 9) = 0$

$b^2 = 16 \vee b^2 = -9$ (spurious)

$b = \pm 4$

$u = 3 + 4i \vee u = -3 - 4i$

$$x^2 = 3 + 4i \vee x^2 = -3 - 4i$$
$$a^2 + 2abi - b^2 = 3 + 4i \vee a^2 + 2abi - b^2 = -3 - 4i$$
$$(a^2 - b^2 = 3 \wedge 2ab = 4) \vee (a^2 - b^2 = -3 \wedge 2ab = -4)$$
$$a = \frac{2}{b} \vee a = -\frac{2}{b}$$
$$4b^{-2} - b^2 = 3 \vee 4b^{-2} - b^2 = -3$$
$$b^4 + 3b^2 - 4 = 0 \vee b^4 - 3b^2 - 4 = 0$$
$$(b^2 + 4)(b^2 - 1) = 0 \vee (b^2 - 4)(b^2 + 1) = 0$$
$$b^2 = -4 \text{ (spurious)} \vee b^2 = 1 \vee b^2 = 4 \vee b^2 = -1 \text{ (spurious)}$$
$$b = 1 \vee b = -1 \vee b = 2 \vee b = -2$$
$$x = 2 + i \vee x = -2 - i \vee x = -1 + 2i \vee x = 1 - 2i$$

f. $z^4 = -7 + 4\sqrt{2}i$

using substitution

$z^4 = u^2 = a + bi$ with $a, b \in \mathbb{R}$ $u^2 = -7 + 4\sqrt{2}i$

$a = \frac{2}{b}\sqrt{2}$

$b^4 - 7b^2 - 8 = 0$

$(b^2 - 8)(b^2 + 1)$

$b^2 = 8 \lor b^2 = -1$ (spurious)

$b = \pm 2\sqrt{2}$

$u = 1 + 2\sqrt{2}i \lor u = -1 - 2\sqrt{2}i$

$$x^2 = 1 + 2\sqrt{2}i \lor x^2 = -1 - 2$$

$$a = \frac{\sqrt{2}}{b} \lor a = -\frac{\sqrt{2}}{b}$$

$$b^4 + b^2 - 2 = 0 \lor b^4 - b^2 - 2 = 0$$

$$(b^2 + 2)(b^2 - 1) \lor (b^2 - 2)(b^2 + 1)$$

$$b^2 = -2 \text{ (spurious)} \lor b^2 = 1 \lor b^2 = 2 \lor b^2 = -1 \text{ (spurious)}$$

$$b = \pm 1 \lor b = \pm\sqrt{2}$$

$$x = \sqrt{2} + i \lor x = -\sqrt{2} - i \lor x = -1 + \sqrt{2}i \lor x = 1 - \sqrt{2}i$$

g. $z^8 = \sqrt{3} + i$

$z^8 = 2e^{\frac{1}{6}\pi + k2\pi}$

$z = 2^{\frac{1}{8}} e^{\frac{1}{48}\pi i + k\frac{12}{48}\pi i}$

$z = 2^{\frac{1}{8}} \cos\left(\frac{1}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{1}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{13}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{13}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{25}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{25}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{37}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{37}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{49}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{49}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{61}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{61}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{73}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{73}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{85}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{85}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{97}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{97}{48}\pi\right)$

h. $z^7 - 2iz^4 - iz^3 - 2 = 0$

$(z^4 - i)(z^3 - 2i) = 0$

$z^4 = i \vee z^3 = 2i$

$z^2 = \pm\sqrt{i} \vee z = 2^{\frac{1}{3}} e^{\frac{1}{6}\pi i + k\frac{4}{6}\pi i}$

$z = \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\ z = -\frac{1}{2}\sqrt{2+\sqrt{2}} + -i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\ z = \frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\ z = -\frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\ z = -2^{-\frac{2}{3}}\sqrt{3} + 2^{-\frac{2}{3}}i$

$\vee\ z = 2^{-\frac{2}{3}}\sqrt{3} + 2^{-\frac{2}{3}}i$

$\vee\ z = -i2^{\frac{1}{3}}$

i. $x^6 + ix^3 + i - 1 = 0$

$x^3 = u = a + bi$ with $a, b \in \mathbb{R}$

$u^2 + iu + i - 1 = 0$

$(u+1)(u+i-1) = 0$

$u = -1 \vee u = 1 - i$

$x^3 = -1 \vee x^3 = 1 - i$

$x = -1$

$\vee\ x = \frac{1}{2} + i\frac{1}{2}\sqrt{3}$

$\vee\ x = \frac{1}{2} - i\frac{1}{2}\sqrt{3}$

$\vee\ x = 1 - i$

$\vee\ x = \sqrt{2}\cos\left(\frac{5}{12}\pi\right) + i\sqrt{2}\sin\left(\frac{5}{12}\pi\right)$

$\vee\ x = \sqrt{2}\cos\left(\frac{13}{12}\pi\right) + i\sqrt{2}\sin\left(\frac{13}{12}\pi\right)$